A NOTE ON GENERALIZED ABSOLUTELY MONOTONE FUNCTIONS AND A BESICOVITCH-TYPE PROBLEM†

BY

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ABSTRACT

We consider in this paper cones of generalized absolutely monotone functions with respect to an infinite sequence of C^∞ $[a,b]$ functions $\{u_i\}_{i=0}^\infty$ such that for all n, $n = 0,1,2,\dots$ $\{u_0, u_1, \dots, u_n\}$ constitutes an Extended Tchebycheff System on $[a,b]$. We find a necessary and sufficient condition for the functions u_i, $i = 0,1,2,\dots$ to generate all the extreme rays in this cone. We conclude by constructing a cone of generalized absolutely monotone functions where the u_i's do not generate all of its extreme rays.

1.

We start by recalling some definitions and results which will be used in the sequel. Let $\{u_i\}_{i=0}^\infty$ be an infinite sequence of functions belonging to $C^\infty[a,b]$ and such that for each n, $n = 0,1,2,\cdots$, $\{u_i\}_{i=0}^n$ constitutes an extended Tchebycheff system on $[a,b]$. With no loss of generality we may assume that

$$u_i(t) = \phi_i(t;a), \qquad i = 0,1,2,\cdots, \tag{1}$$

where

$$\phi_i(t;x) = \begin{cases} w_0(t)\displaystyle\int_x^t w_1(\xi_1)\int_x^{\xi_1} w_2(\xi_2)\cdots\int_x^{\xi_{i-1}} w_i(\xi_i)\,d\xi_i\cdots d\xi_1, & x \leqq t \leqq b \\ 0, \quad a \leqq t < x \end{cases} \tag{2}$$

for $i = 0,1,2,\cdots$, $a \leqq x \leqq b$, $w_k > 0$, and $k = 0,1,2,\cdots$. (See [4].)

DEFINITION 1. A function f defined on (a,b) is said to be *convex with respect*

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to the Tchebycheff system $\{u_i\}_{i=0}^{n}$ if for every set of $n+2$ points $\{t_i\}_{i=0}^{n+1}$ satisfying $a < t_0 < t_1 < \cdots < t_{n+1} < b$, the following determinant inequality holds:

$$(3)\qquad U\begin{pmatrix} u_0, u_1, \cdots, u_n, f \\ t_0, t_1, \cdots, t_n, t_{n+1} \end{pmatrix} = \begin{vmatrix} u_0(t_0) & \cdots & u_0(t_{n+1}) \\ \vdots & & \vdots \\ u_n(t_0) & \cdots & u_n(t_{n+1}) \\ f(t_0) & \cdots & f(t_{n+1}) \end{vmatrix} \geqq 0.$$

The set of the convex functions with respect to a given Tchebycheff system is a convex cone denoted by $C(u_0, u_1, \cdots, u_n)$.

It is proved in [3] that $f \in C^+ \cap [\bigcap_{i=0}^{\infty} C(u_0, u_1, \cdots, u_i)]$, where C^+ denotes the cone of the non-negative functions defined on (a, b), if and only if

$$(4)\qquad \begin{aligned} &(L_{-1}f)(t) = f(t) \geqq 0 \\ &(L_i f)(t) = (D_i D_{i-1} \cdots D_0 f)(t) \geqq 0, \qquad i = 0, 1, 2, \cdots, \qquad a < t < b \end{aligned}$$

where

$$(D_k f)(t) = \frac{d}{dt} \frac{f(t)}{w_k(t)}.$$

It is also shown that if $f \in C_A = C^+ \cap [\bigcap_{i=0}^{\infty} C(u_0, u_1, \cdots, u_i)]$ then the following Taylor-type formula holds:

$$(5)\qquad f(t) = \int_a^b \phi_n(t; x)(L_n f)(x)dx + \sum_{i=0}^{n} \frac{(L_{i-1}f)(a+)}{w_i(a)} u_i(t).$$

This formula is an extreme rays representation for functions in the cone $C^+ \cap [\bigcap_{i=0}^{n} C(u_0, u_1, \cdots, u_i)]$, (refer to [3]).

A necessary and sufficient condition for a function $f \in C_A$ to admit a Taylor-type representation

$$(6)\qquad f(t) = \sum_{i=0}^{\infty} a_i u_i(t)$$

where

$$a_i = \frac{(L_{i-1}f)(a+)}{w_i(a)}$$

is that for every t, $a < t < b$, there exists s, $t < s < b$, such that

$$(7)\qquad \lim_{i \to \infty} \frac{u_i(t)}{u_i(s)} = 0.$$

(Refer to [1].)

If we restrict ourselves to the cone $C_A \cap B$, where B denotes the set of bounded functions on (a, b) then condition (7) should be replaced by

$$\lim_{i \to \infty} \frac{u_i(t)}{u_i(b)} = 0, \qquad \forall t,\ a < t < b. \tag{7'}$$

In this paper we give another necessary and sufficient condition for (6) to hold for generalized absolutely monotone functions in (a, b). We use here the condition found by Amir and Ziegler in [1]. With no loss of generality we may assume that $w_0(t) = 1$ since $f \in C(u_0, u_1, \cdots, u_n)$ if and only if $f/w_0 \in C(1, u_1/w_0, \cdots, u_n/w_0)$ and the representation (6) holds for f if and only if $f(t)/w_0(t) = \sum_{i=0}^{\infty} a_i(u_i(t)/w_0(t))$.

2. The generalized Besicovitch problem

In [2] Besicovitch shows that, given a function f, positive and continuous in $[0, b)$ and $f(t) \to \infty$ as $t \to b$, there exists a power series $p(t) = \sum_{n=0}^{\infty} a_n t^n$ with $a_n \geqq 0$ such that $p(t) < f(t)$ for all $0 \leqq t < b$ and $p(t) \to \infty$ as $t \to b$.

Consider the following problem.

PROBLEM 2. Let f be defined, continuous and positive in $[a, b)$ and $f(t) \to \infty$ as $t \to b$, and let $\{u_i\}_{i=0}^{\infty}$ be a sequence of functions defined by (1) and (2). Does there exist a series

$$p(t) = \sum_{i=0}^{\infty} a_i u_i(t) \tag{8}$$

with $a_i \geqq 0$ such that $p(t) < f(t)$ in (a, b) and $p(t) \to \infty$ as $t \to b$?

CLAIM 3. The answer is affirmative if and only if (7') holds.

PROOF. Without loss of generality we may assume that $u_i(b) = 1$, $i = 0, 1, 2, \cdots$ Condition (7') should be replaced by:

$$\lim_{i \to \infty} u_i(t) = 0, \qquad a \leqq t < b. \tag{9}$$

Suppose (9) does not hold for some t_0; since $\{u_i(t)/u_i(b)\}_{i=0}^{\infty}$ is a decreasing sequence (see [1]), there exists an $\varepsilon > 0$ such that

$$u_i(t_0) > \varepsilon, \qquad i = 0, 1, 2, \cdots. \tag{10}$$

From $p(t_0) < f(t_0)$ we have: $\sum_{i=0}^{\infty} a_i u_i(t_0) < \infty$, but by (10) we have $\sum_{i=0}^{\infty} a_i < \infty$, that is, $\sum_{i=0}^{\infty} a_i u_i(b) < \infty$.

Now $p(t) = \sum_{i=0}^{\infty} a_i u_i(t)$ is a monotone function, $p(t) < p(b)$ and hence $\lim_{t\to b} p(t) < \infty$. The proof of the sufficiency follows in the same line as the solution of Besicovitch problem [2]. We give the proof for the sake of completeness.

LEMMA 4. *Let f be a function defined on $[a,b)$ satisfying the conditions of Problem 2, and $\{u_i\}_0^{\infty}$ a sequence of non-negative functions such that:*

(i) $u_i(b) = 1$, $i = 0,1,2,\cdots$,

(ii) *for all t, $\{u_i(t)\}_{i=0}^{\infty}$ is a decreasing sequence, and*

(iii) *for all t, $\lim_{i\to\infty} u_i(t) = 0$.*

There exists an integer n_1 such that $f - u_{n_1}$ satisfies the condition of the problem.

PROOF. Denote $m = \min_{a \leqq t < b} f(t) > 0$. Let t_0 be such that $f(t) > 1$ for all $t \geqq t_0$ and n_1 be such that $u_{n_1}(t_0) < m$. It is readily seen that $f_1 = f - u_{n_1}$ satisfies the condition of problem 2.

We continue the proof of the sufficiency. By Lemma 4 there exists an increasing sequence of natural numbers: $n_1 < n_2 < \cdots$ such that

$$f_1 = f - u_{n_1} > 0$$

$$f_2 = f_1 - u_{n_2} > > 0$$

$$f_k = f_{k-1} - u_{n_k} = f - [u_{n_1} + u_{n_2} + \cdots + u_{n_k}] > 0.$$

Hence $p(t) = \sum_{i=0}^{\infty} u_{n_i}(t) \leqq f(t)$ and $\lim_{t\to b} p(t) = \infty$. Note that $\{u_0, u_1, \cdots, u_n\}$ need not be a Tchebycheff system; however for a sequence $\{u_i\}_{i=0}^{\infty}$ defined by (1) and (2) we have the following theorem.

THEOREM 5. *The polynomials $\{u_i\}_{i=0}^{\infty}$ generate all the extreme rays in $C_A \cap B$ if and only if there exists a series $p(t) = \sum_{i=0}^{\infty} a_i u_i(t)$ with $a_i \geqq 0$, $i = 0,1,2,\cdots$, converging for $a \leqq t < b$ and such that $\lim_{t\to b} p(t) = \infty$.*

3.

We conclude by showing that there exists a sequence $\{u_i\}_{i=0}^{\infty}$ defined by (1) and (2) such that $\lim_{i\to\infty} u_i(t)/u_i(b) > 0$.

Let $\{\tilde{w}_k\}_{k=0}^{\infty}$ be an infinite sequence of positive $C^{\infty}[a,b]$ functions such that for every function f, defined and integrable on $[a,b]$ and continuous in $[a, a+\varepsilon]$ $(\varepsilon > 0)$:

$$\lim_{k\to\infty} \int_a^b f(t)\tilde{w}_k(t)\,dt = f(a).$$

Define $u_0 = 1$ and

$$\phi_0(t; x) = \begin{cases} 0 & a \leqq t < x \\ 1 & x \leqq t \leqq b. \end{cases}$$

Let t_0 be a fixed number, $a < t_0 < b$, and let $0 < \varepsilon < \frac{1}{2}$. Choose k_1, and denote $w_1 = \tilde{w}_{k_1}$, such that

$$u_1(t_0) = \int_a^b \phi_0(t_0; x) w_1(x)\, dx \geqq \phi_0(t_0; a) - \frac{\varepsilon}{2}$$

and

$$u_1(t) = \int_a^b \phi_0(t; x) w_1(x) dx \leqq \phi_0(b; a) + \frac{\varepsilon}{2}.$$

We now define $\phi_1(\cdot\,; x)$ by

$$\phi_0(t;x) = \begin{cases} 0 & a \leqq t < x \\ \int_x^t w_1(\xi_1) d\xi_1 & x \leqq t \leqq b. \end{cases}$$

Suppose that $\phi_0(\cdot; x)$, $\phi_1(\cdot; x), \cdots, \phi_{n-1}(\cdot; x)$ has been defined such that

$$u_i(t_0) \geqq u_0(t_0) - \left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^i}\right]$$

$$u_i(t) \leqq u_0(t) + \left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^i}\right]$$

for $i = 1, 2, \cdots, n-1$.

Choose k_n, and denote $w_n = \tilde{w}_{k_n}$, such that

$$u_n(t_0) = \int_a^b \phi_{n-1}(t_0; x) w_n(x) dx \geqq \phi_{n-1}(t_0; a) - \frac{\varepsilon}{2^n}$$

$$\geqq u_0(t_0) - \left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^n}\right] > 1 - \varepsilon$$

and

$$u_n(t) = \int_a^b \phi_{n-1}(t; x) w_n(x)\, dx \leqq \phi_{n-1}(b; a) + \frac{\varepsilon}{2^n}$$

$$\leqq u_0(t) + \left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^n}\right] < 1 - \varepsilon.$$

Hence

$$\frac{u_n(t_0)}{u_n(b)} \geq \frac{1-\varepsilon}{1+\varepsilon} \geq \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}.$$

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